Reconstruction of the Scalar-Tensor Lagrangian from a ΛCDM Background and Noether Symmetry

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We consider scalar-tensor theories and reconstruct their potential $U(\Phi)$ and coupling $F(\Phi)$ by demanding a background ΛCDM cosmology. In particular we impose a background cosmic history $H(z)$ provided by the usual flat ΛCDM parameterization through the radiation ($w_{eff} = 1/3$), matter ($w_{eff} = 0$) and deSitter ($w_{eff} = -1$) eras. The cosmological dynamical system which is constrained to obey the ΛCDM cosmic history presents five critical points in each era, one of which corresponding to the standard General Relativity (GR). In the cases that differ from GR, the reconstructed coupling and potential are of the form $F(\Phi) \sim \Phi^2$ and $U(\Phi) \sim F(\Phi)^m$ where m is a constant. This class of scalar tensor theories is also theoretically motivated by a completely independent approach: imposing maximal Noether symmetry on the scalar-tensor Lagrangian. This approach provides independently: *i*) the form of the coupling and the potential as $F(\Phi) \sim \Phi^2$ and $U(\Phi) \sim F(\Phi)^m$, *ii*) a conserved charge related to the potential and the coupling and *iii*) allows the derivation of exact solutions by first integrals of motion.

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I. INTRODUCTION

There is accumulating observational evidence based mainly on Type Ia supernovae standard candles [1] and also on standard rulers [2, 3] that the universe has entered a phase of accelerating expansion at a recent cosmological timescale. Such an expansion implies the existence of a repulsive pressure on cosmological scales which counterbalances the gravitational attraction of matter on these scales giving rise to an overall accelerating behavior. There have been several theoretical approaches (see [4, 5] for a review) to understand this phenomenon. The simplest of such approaches assumes the existence of a positive cosmological constant which is small enough to have started dominating the universe at recent times. The predicted cosmic expansion history, in this case, (assuming flatness) is

$$H(z)^2 = \left(\frac{\dot a}{a}\right)^2 = H_0^2\left[\Omega_{0\mathrm{m}}(1+z)^3 + \Omega_{0r}(1+z)^4 + \Omega_\Lambda\right] \tag{1.1}$$

where $\Omega_{0r} = \frac{\rho_r}{\rho_{crit}} \simeq 10^{-4}$ is the energy density of radiation today normalized over the critical density for flatness ρ_{crit}. Also $\Omega_{0\mathrm{m}} = \frac{\rho_m}{\rho_{crit}} \simeq 0.3$ is the normalized present matter density and $\Omega_\Lambda = 1 - \Omega_{0\mathrm{m}} - \Omega_{0r}$ is the normalized energy density due to the cosmological constant. This model provides an excellent fit to the cosmological observational data [2] and has the additional bonus of simplicity due to a single free parameter. Despite its simplicity and good fit to the data, such a model fails to explain why the cosmological constant, so unnaturally small dominates the universe at recent cosmological times. This problem is known as the *coincidence problem*. Furthermore, there is no urgent theoretical reason implying $\Omega_\Lambda \simeq 0.7$ and $\Omega_{0\mathrm{m}} \simeq 0.3$ at present time so also a *fine tuning problem* has to be taken into account.

In the effort to address these problems, several models, which essentially can be grouped in two classes, have been proposed: The first class assumes that General Relativity (GR) is valid at cosmological scales and attributes the accelerating expansion to a *dark energy* component which has repulsive gravitational properties due to its negative pressure. The role of dark energy is usually played by a scalar field minimally coupled to gravity called *quintessence* [6]. Alternatively, the role of dark energy can be played by various perfect fluids (eg Chaplygin gas [7]) topological defects [8], holographic dark energy [9] etc.

The second class of models attributes the accelerating expansion to modifications and extensions of GR which converts gravity to a repulsive interaction at late times and on cosmological scales. Examples of this class of models include scalar-tensor theories [10, 11], $f(R)$ extended gravity theories [12], braneworld models [13] etc. An advantage of models in this class is that they naturally allow [14, 15] a super-accelerating expansion of the universe where the effective dark energy equation of state $w = \frac{p}{\rho}$ crosses the phantom divide line $w = -1$. Such a crossing is consistent with current cosmological data [16, 17].

A representative model of the second class is provided by scalar-tensor theories of gravity. In these theories, the Newton constant is obtained by dynamical properties expressed through the coupling $F(\Phi)$. Dynamics is given

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by the Lagrangian density [18, 19, 20, 21]

$$\mathcal{L} = \frac{F(\Phi)}{2}\ R - \frac{1}{2}\ \epsilon\ g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - U(\Phi) + \mathcal{L}_{\rm m}[\psi_{\rm m}; g_{\mu\nu}] \tag{1.2}$$

where $\mathcal{L}_{\rm m}[\psi_{\rm m}; g_{\mu\nu}]$ represents matter fields approximated by a pressureless perfect fluid in the dust dominated regime. We have set $8\pi G = 1$ and $\epsilon = \pm 1$ for standard scalar and phantom fields respectively, i.e. we have negative kinetic energy for the scalar degree of freedom. Note, however that negative energy needs $\epsilon = -1$, but there also exist positive energy configurations with $\epsilon = -1$. This happens due to the fact that the kinetic term of the actual spin-0 degree of freedom does not come only from the obvious $(\partial_\mu\Phi)^2$, but also from the cross term $F(\Phi)R$, since R involves second derivatives (see [21] for details). As discussed below, for $\epsilon = 0$ and $\Phi \to R$ the Lagrangian (1.2) can also describe $f(R)$ generalizations of GR.

In the present study, we reconstruct the potential $U(\Phi)$ and the coupling $F(\Phi)$. Instead of specifying various forms of $U(\Phi)$ and $F(\Phi)$ and finding the corresponding cosmological dynamics, we specify the cosmological dynamics to that of the ΛCDM cosmology and search for possible corresponding forms of $U(\Phi)$ and $F(\Phi)$. The original method for the reconstruction of scalar-tensor theories from a given cosmic history $H(z)$ was introduced in Ref. [15] and applied to specific cases of late cosmic history in Ref. [14, 21]. Our reconstruction approach is different in two aspects:

- We use a dynamical system formalism and find the critical points that determine the generic evolution of the system.
- We start the reconstruction from the radiation era rather than focusing only at late times through the acceleration epoch.

In particular, we consider the general dynamical system for scalar-tensor theories and study the dynamics of $U(\Phi)$ and $F(\Phi)$ using as input a ΛCDM cosmic expansion history. Our study is performed both analytically (using the critical points and their stability) and numerically by explicitly solving the dynamical system.

Since our reconstruction assumes a fixed cosmic history background the stability of the critical points we find should be interpreted with care. The fixing of the cosmic history has eliminated perturbations of $H(z)$ but has allowed for perturbations in the forms of the coupling $F(\Phi)$ and the potential $U(\Phi)$. In the usual stability approach the coupling $F(\Phi)$ and potential $U(\Phi)$ are fixed and perturbations are allowed in the cosmic history $H(z)$ to determine the stability of the phase space trajectories. Since the later stability approach is more physical (but does not lead to a reconstruction) we focus on the actual values of the critical points and interpret their stability only as a test of the corresponding numerical evolution of the same system. Thus, all the critical points we find are assumed equally important cosmologically, independent of their stability.

The structure of the paper is the following: In the next section we derive the dynamical system for the cosmological dynamics of scalar-tensor theories. Using as input a particular cosmic history $H(z)$ (eg ΛCDM), we show how this system can be transformed so that its solution provides the dynamics and the functional form of $U(\Phi)$ and $F(\Phi)$. We also study the dynamics of this transformed system analytically by deriving its critical points during the three eras of the cosmic background history (radiation, matter and deSitter). We find that the cosmological dynamical system constrained to obey the ΛCDM cosmic history has five critical points in each era, one of which corresponds to GR. In Sect. III, we use the solution of the above system to reconstruct the cosmological evolution and functional form of the coupling and potential, which are of the form $F(\Phi) \sim \Phi^2$ and $U(\Phi) \sim F(\Phi)^m$ where m is an arbitrary constant. We show that such forms are also motivated by a completely independent approach ie by imposing Noether symmetry on the scalar-tensor Lagrangian [22]. We also demonstrate the agreement between the analytical and numerical results of our reconstruction scheme. Finally, in Sect.IV, we conclude, summarize and refer to future prospects of this work.

II. DYNAMICS OF SCALAR-TENSOR COSMOLOGIES

Let us consider the action (1.2) describing the dynamics of Scalar Tensor theories in the Jordan frame. In the context of flat Friedman-Robertson-Walker (FRW) universes, the metric is homogeneous and isotropic *i.e.*

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\mathbf{x}^2 \tag{2.1}$$

Variation of the action (1.2) with respect to the metric leads to the following dynamical equations which are the generalized Friedman equations [20, 21, 23]

$$3FH^2 \ = \ \rho_{\rm m} + \rho_r + \frac{1}{2}\epsilon\dot{\Phi}^2 - 3H\dot{F} + U \tag{2.2}$$

$$-2F\dot{H} \ = \ \rho_{\rm m} + \frac{4}{3}\rho_r + \epsilon\dot{\Phi}^2 + \ddot{F} - H\dot{F} \tag{2.3}$$

and variation with respect to Φ gives the Klein-Gordon equation:

$$\epsilon(\ddot{\Phi} + 3H\dot{\Phi}) = 3F(\Phi)_{,\Phi}(\dot{H} + 2H^2) - U(\Phi)_{,\Phi} \tag{2.4}$$

where we have assumed the presence of perfect fluids $\rho_{\rm m}$, ρ_r representing the matter and radiation energy densities which are conserved according to

$$\dot{\rho}_{\rm m} + 3H\rho_{\rm m} = 0\,, \tag{2.5}$$

$$\dot{\rho}_{\rm r} + 4H\rho_{\rm r} = 0\,. \tag{2.6}$$

The background equations in eqs. (2.2) and (2.3) can be rewritten in a more convenient form, which is easier to confront with observations (see for example Ref. [24]):

$$3F_0H^2 = \rho_{DE} + \rho_{\rm m} + \rho_r, \tag{2.7}$$

$$-2F_0\dot{H} = \rho_{DE} + p_{DE} + \rho_{\rm m} + \frac{4}{3}\rho_r \tag{2.8}$$

where we have set

$$\rho_{DE} = \frac{1}{2}\epsilon\dot{\Phi}^2 - 3H^2(F-F_0) - 3H\dot{F} + U \tag{2.9}$$

$$p_{DE} = \frac{\epsilon}{2}\dot{\Phi}^2 + \ddot{F} + 2H\dot{F} - U - (2\dot{H} + 3H^2)(F_0 - F) \tag{2.10}$$

and the subscript "0" denotes present day values. The function ρ_{DE} defined in this way can be shown to satisfy the usual energy conservation equation:

$$\dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = 0 \tag{2.11}$$

where the DE equation of state is defined as

$$w_{DE} \equiv \frac{p_{DE}}{\rho_{DE}} = -1 + \frac{\epsilon\dot{\Phi}^2 + \ddot{F} - H\dot{F} - 2\dot{H}(F_0 - F)}{\frac{1}{2}\epsilon\dot{\Phi}^2 - 3H^2(F-F_0) - 3H\dot{F} + U} \tag{2.12}$$

By using eqs. (2.7) and (2.8) we can express the equation of state w_{DE} as

$$w_{DE} = -\frac{3E(z) - (1+z)(dE(z)/dz)}{3E(z) - 3\Omega_m^{(0)}(1+z)^3} \tag{2.13}$$

where we have set $E(z) \equiv H^2(z)/H_0^2$. This relation is exactly the same as in standard Einstein gravity [5] and therefore we can constrain w_{DE} from Type Ia supernovae observations in the same way.

In order to study the cosmological dynamics implied by equations (2.2), (2.3) and (2.4) we express them as a dynamical system of first order differential equations. To achieve this, let us first write (2.2) in dimensionless form as

$$1 = \frac{\rho_{\rm m}}{3FH^2} + \frac{\rho_r}{3FH^2} + \epsilon\frac{\Phi'^2}{6F} + \frac{U}{3FH^2} - \frac{F'}{F} \tag{2.14}$$

where

$$' = \frac{d}{d{\rm ln a}} \equiv \frac{d}{dN} = \frac{1}{H}\frac{d}{dt} \tag{2.15}$$

Let us now define the dimensionless variables $x_1, ..., x_4$ as

$$x_1 = -\frac{F'}{F}\,, \tag{2.16}$$

$$x_2 = \frac{U}{3FH^2}\,, \tag{2.17}$$

$$x_3^2 = \frac{\Phi'^2}{6F}\,, \tag{2.18}$$

$$x_4 = \frac{\rho_{\rm r}}{3FH^2} = \Omega_r\,. \tag{2.19}$$

where we can associate x_4 with Ω_r and $x_1 + x_2 + \epsilon x_3^2 \equiv \Omega_{DE}$ with curvature dark energy (dark gravity). Defining also $\Omega_m \equiv \frac{\rho_{\rm m}}{3FH^2}$ we can write equation (2.14) as

$$\Omega_m = 1 - x_1 - x_2 - \epsilon x_3^2 - x_4 \tag{2.20}$$

Let us now use (2.15) to express (2.3) as

$$\frac{H'}{H} = -\frac{\rho_{\rm m}}{2FH^2} - \frac{2}{3}\frac{\rho_r}{FH^2} - \epsilon\frac{\Phi'^2}{2F} - \frac{F''}{2F} - \frac{H'F'}{2FH} + \frac{F'}{2F} \tag{2.21}$$

or

$$x_1' = 3 - 2x_1 - 3x_2 + x_4 + 3\epsilon x_3^2 + x_1^2 + 2\frac{H'}{H} - x_1\frac{H'}{H} \tag{2.22}$$

Differentiating x_4 of (2.19) with respect to N, we have

$$x_4' = \frac{\rho_r'}{3FH^2} - \frac{\rho_r}{3FH^2}\frac{F'}{F} - \frac{2\rho_r}{3FH^2}\frac{H'}{H} \tag{2.23}$$

or

$$x_4' = -4x_4 + x_4x_1 - 2x_4\frac{H'}{H} \tag{2.24}$$

where we have used (2.6). Similarly, differentiating (2.17) with respect to N, we find

$$x_2' = x_2\left[x_1(1-m) - 2\frac{H'}{H}\right] \tag{2.25}$$

where

$$m \equiv \frac{U_{,\Phi}/U}{F_{,\Phi}/F} \tag{2.26}$$

and $_{,\Phi}$ implies derivative with respect to Φ. Finally differentiating (2.18) with respect to N and using (2.4), one finds

$$\epsilon(x_3^2)' = \epsilon x_3^2x_1 - 6\epsilon x_3^2 - 2x_1 + mx_2x_1 - 2\epsilon x_3^2\frac{H'}{H} - x_1\frac{H'}{H} \tag{2.27}$$

The dynamical system (2.22), (2.24), (2.25), (2.27) describes the cosmological dynamics of Scalar-Tensor theories. In two special limits, it can be transformed into the dynamical systems obtained in $f(R)$ theories and Quintessence respectively [25]. However, we have to say that there is one more degree freedom in scalar-tensor theories, compared to $f(R)$ theories, since here we have two arbitrary functions, i.e. $F(\Phi)$ and $U(\Phi)$. In general, we should add a further parameter n which could be related to $F_{,\Phi}$ and $F_{,\Phi\Phi}$, beside the above m. In fact, if we do not try to reconstruct the function $F(\Phi)$, such a function can be fixed a priori and the corresponding parameter n would be, for example, $F_{,\Phi}/F$, or some similar condition. In such a case $H(N)$ would not be fixed as in our reconstruction approach but would have to be determined by the autonomous system. This is the approach followed in [26] where F_{Φ}/F is not present as a variable in the autonomous system since $F(\Phi)$ is fixed a priori.

The parameters in [26] are the power law of the fixed potential (there called n) and ξ, the coupling $F(\Phi) = \xi\Phi^2$. In that case, $H(N)$ and its perturbations are allowed to vary. However, in a reconstruction approach where $H(N)$ is fixed a priori, while $F_{,\Phi}/F$ (and its perturbations) are allowed to vary, $F(\Phi)$ can be reconstructed from the autonomous system. Thus, in our reconstruction approach where $H(N)$ is fixed, n is no more a parameter and it is allowed to evolve. In this sense, the present approach is more general than that presented in [26]. Finally, as we shall see below, the reconstructed function $F(\Phi)$ and $U(\Phi)$ can be independently obtained asking for a sort of "first principle", the existence of a Noether Symmetry. Since the results coincide (the form of F and U are the same in both the approaches), we are confident that the presented method is consistent.

Considering again the $f(R)$-theories, by setting $U = \frac{FR-f}{2}$ [27] and using the transformations

$$\begin{aligned} x_1 &\rightarrow \tilde{x}_1 &(2.28)\\ x_2 &\rightarrow \tilde{x}_2 + \tilde{x}_3 &(2.29)\\ x_4 &\rightarrow \tilde{x}_4 &(2.30)\\ \frac{H'}{H} &\rightarrow \tilde{x}_3 - 2 &(2.31)\\ \Phi &\rightarrow R &(2.32)\\ \epsilon &\rightarrow 0 &(2.33)\end{aligned}$$

one recovers the dynamical system of $f(R)$ theories (eqs (2.15), (2.17) and (2.21) in Ref. [28]), where the *tilded* ($\tilde{}$) quantities are the ones defined for $f(R)$ theories:

$$\tilde{x}_1' = -1 - \tilde{x}_3 - 3\tilde{x}_2 + \tilde{x}_1^2 + \tilde{x}_4 \qquad (2.34)$$

$$\tilde{x}_2' = -\tilde{x}_3' - 2\tilde{x}_3(\tilde{x}_3 - 2) - \tilde{x}_2(2\tilde{x}_3 - \tilde{x}_1 - 4) \qquad (2.35)$$

$$\tilde{x}_4' = -2\tilde{x}_3\ \tilde{x}_4 + \tilde{x}_1\ \tilde{x}_4 \qquad (2.36)$$

On the other hand, the following set of transformations gives the autonomous system for Quintessence (see eqs (175) and (176) in Ref. [5]):

$$\begin{aligned} x_1 &\rightarrow 0 &(2.37)\\ x_2 &\rightarrow y^2 &(2.38)\\ x_3^2 &\rightarrow x^2 &(2.39)\\ x_4 &\rightarrow 0 &(2.40)\end{aligned}$$

with dynamical equations

$$x' = -3x + \frac{\sqrt{6}}{2}\epsilon\lambda y^2 + \frac{3}{2}x(\epsilon x^2 + 1 - y^2) \qquad (2.41)$$

$$y' = -\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y(\epsilon x^2 + 1 - y^2) \qquad (2.42)$$

and $\lambda = -U_{,\Phi}/U$.

It is worth noting that the results of our analysis do not rely on the use of any particular form of $H(z)$. They only require that the universe goes through the radiation era (high redshifts), matter era (intermediate redshifts) and acceleration era (low redshifts). The corresponding total effective equation of state

$$w_{eff} = -1 - \frac{2}{3}\frac{H'(N)}{H(N)} \qquad (2.43)$$

is

$$\begin{aligned} w_{eff} &= \frac{1}{3} \quad \text{Radiation Era}\\ w_{eff} &= 0 \quad \text{Matter Era} \qquad (2.44)\\ w_{eff} &= -1 \quad \text{deSitter Era}\end{aligned}$$

For the sake of definiteness however, we will assume a specific form for $H(z)$ corresponding to a ΛCDM cosmology (1.1) which, in terms of N, takes the form

$$H(N)^2 = H_0^2\left[\Omega_{0\mathrm{m}}e^{-3N} + \Omega_{0r}e^{-4N} + \Omega_\Lambda\right] \qquad (2.45)$$

where $N \equiv \ln a = -\ln(1+z)$ and $\Omega_\Lambda = 1 - \Omega_{0\mathrm{m}} - \Omega_{0r}$. Also, we can use (2.45) to find $\frac{H'(N)}{H(N)}$, a quantity needed in the dynamical system, as

$$\frac{H'(N)}{H(N)} = \frac{-3\,\Omega_{0m}e^{-3N} - 4\,\Omega_{0r}e^{-4N}}{2\,(1 - \Omega_{0m} - \Omega_{0r} + \Omega_{0m}e^{-3N} + \Omega_{0r}e^{-4N})} \qquad (2.46)$$

The crucial generic properties of $\frac{H'(N)}{H(N)}$ are their values at the radiation, matter and deSitter eras:

$$\begin{aligned} \frac{H'(N)}{H(N)} &= -2 \quad N < N_{rm} &(2.47)\\ \frac{H'(N)}{H(N)} &= -\frac{3}{2} \quad N_{rm} < N < N_{m\Lambda} &(2.48)\\ \frac{H'(N)}{H(N)} &= 0 \quad N > N_{m\Lambda} &(2.49)\end{aligned}$$

where $N_{rm} \simeq -\ln\frac{\Omega_{0\mathrm{m}}}{\Omega_{0r}}$ and $N_{m\Lambda} \simeq -\frac{1}{3}\ln\frac{\Omega_\Lambda}{\Omega_{0\mathrm{m}}}$ are the N values for the radiation-matter and matter-deSitter transitions. For $\Omega_{0\mathrm{m}} = 0.3$, $\Omega_{0r} = 10^{-4}$ we have $N_{rm} \simeq -8$, $N_{m\Lambda} \simeq -0.3$. The transition between these eras is model dependent but rapid and it will not play an important role in our analysis.

It is straightforward to study the dynamics of the system (2.22), (2.24), (2.25), (2.27) by finding the critical points and their stability in each one of the three eras. Notice that even though this dynamical system is not autonomous at all times, it can be approximated as such during the radiation, matter and deSitter eras when $\frac{H'(N)}{H(N)}$ is approximately constant. The critical points and their eigenvalues are shown in Table I. An interesting feature to observe in Table I is that in each era there are five critical points but only one of them is a stable "attractor" for a given value of m. Also, remembering that

TABLE I: The critical points of the system (2.22), (2.24), (2.25), (2.27) and their eigenvalues in each one of the three eras (rad. era $N < -\ln\frac{\Omega_{0m}}{\Omega_{0r}}$, matter era $-\ln\frac{\Omega_{0m}}{\Omega_{0r}} < N < -\frac{1}{3}\ln\frac{\Omega_\Lambda}{\Omega_{0m}}$, deSitter era $N > -\frac{1}{3}\ln\frac{\Omega_\Lambda}{\Omega_{0m}}$).

Era	**CP**	x_1	x_2	x_3^2	x_4	Ω_m	Ω_{DE}	**Eigenvalues**
	R_1	2	0	-1	0	0	1	(2,3,1,6-2m)
	R_2	1	0	0	0	0	1	(1,2,-1,5-m)
Radiation	R_3	-1	0	0	0	2	-1	(-1,-2,-3,3+m)
$w_{eff}=\frac{1}{3}$	R_4	$\frac{4}{-1+m}$	$\frac{15-8m+m^2}{3(m-1)^2}$	$\frac{2(m-5)m}{3(m-1)^2}$	0	0	1	$(\frac{4}{m-1}, \frac{m+3}{m-1}$, see [a])
	R_5	0	0	0	1	0	0	(1,-1,-2,4)
	M_1	2	0	-1	0	0	1	(1,2,1/2,5 $-$ 2m)
	M_2	3/2	0	-1/2	0	0	1	(1/2,3/2,-1/2,$-3/2(m-3)$)
Matter	M_3	0	0	0	0	0	0	(-1,-3/2,-2,3)
$w_{eff}=0$	M_4	$\frac{3}{m-1}$	$\frac{15-11m+2m^2}{4(m-1)^2}$	$\frac{1-9m+2m^2}{4(m-1)^2}$	0	0	1	$(\frac{4-m}{m-1}, \frac{3}{m-1}$, see [b])
	M_5	1	0	-1/4	1/4	0	3/4	(1,-1/2,-1,4 $-$ m)
	Λ_1	2	0	-1	0	0	1	(-2,-1,-1,2 $-$ 2m)
	Λ_2	3	0	-2	0	0	1	(-1,0,1,3 $-$ 3m)
deSitter	Λ_3[c]	3	0	-2	0	0	1	(-1,0,1,3 $-$ 3m)
$w_{eff}=-1$	Λ_4	0	1	0	0	0	1	$\left(-4,-3,-\frac{\sqrt{24m+1}+5}{2},\frac{\sqrt{24m+1}-5}{2}\right)$
	Λ_5	4	0	-4	1	0	0	(1,1,2,4 $-$ 4m)

[a] the other two eigenvalues are: $-\frac{3m+\sqrt{8m^3-63m^2+118m+1}-11}{2(m-1)}, \frac{-3m+\sqrt{8m^3-63m^2+118m+1}+11}{2m-2}$

[b] the other two eigenvalues are: $-\frac{7m+\sqrt{48m^3-263m^2+358m+1}-19}{4(m-1)}, \frac{-7m+\sqrt{48m^3-263m^2+358m+1}+19}{4m-4}$

[c] Notice that Λ_2 and Λ_3 are degenerate

the positivity of the energy of the (helicity zero) scalar partner of the graviton, i.e. the positivity of the kinetic energy of the scalar field (see Ref. [11]) is expressed by:

$$\phi'^2 = \frac{3}{4}\left(\frac{F'}{F}\right)^2 + \frac{\epsilon\Phi'^2}{2F} > 0 \qquad (2.50)$$

then we have in dimensionless variables

$$\frac{x_1^2}{4} + x_3^2 > 0 \qquad (2.51)$$

All points of Table I satisfy (2.51) except R_4 and M_4. In order for R_4 to satisfy (2.51), we must have $m < 2$ or $m > 3$ while for M_4 it is $m < 2$ or $m > 5/2$. Therefore the allowed region for m, so that we have critical points that are physical ($\frac{x_1^2}{4} + x_3^2 > 0$), is

$$m \leq 2 \quad or \quad m \geq 3 \qquad (2.52)$$

Now, regarding the "attractor" behavior of the system in each era separately, we see that in:

- Radiation Era: R_3 is an "attractor" for $m < -3$ while R_4 is an "attractor" for $-3 < m < 1$;
- Matter Era: M_4 is an "attractor" for $m < 1$;
- deSitter Era: Λ_1 is an "attractor" for $m > 1$ and Λ_4 is an "attractor" for $m < 1$.

As discussed in the introduction, the significance of the "attracting" nature of the critical points is limited because the reconstruction method we are using has forced us to eliminate physical perturbations of $H(N)$ and has introduced unphysical perturbations of the functional forms of the potential $U(\Phi)$ and coupling $F(\Phi)$. However, we can use the "attractor" critical points as a prediction for the numerical evolution of our reconstruction dynamical system.

A particularly interesting feature of the critical points of Table I is that in all the cases that differ from GR, the expansion rate in each era is induced by dark energy ($\Omega_{DE} = 1$) which implies that the scalar field Φ could also play the role of dark matter if it is found to have the proper perturbation properties at early times. We postpone the analysis of such perturbations for a future study.

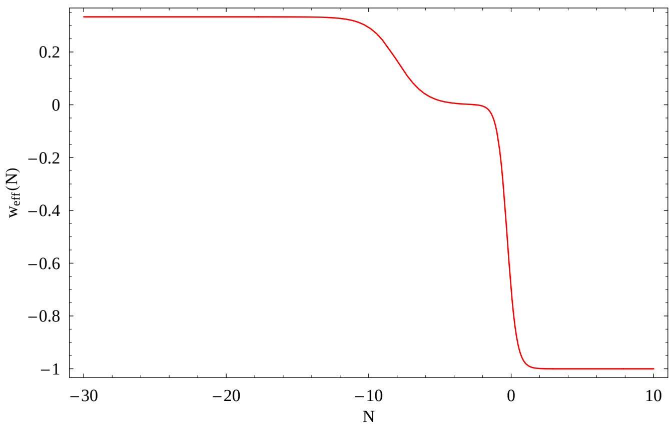

FIG. 1: The effective equation of state $w_{eff}(N)$ imposed on the dynamical system (obtained from (2.43) using (2.45)).

To confirm the dynamical evolution implied by the "attractors" of Table I, we have performed a numerical analysis of the dynamical system (2.22), (2.24), (2.25) and (2.27) using the ansatz (2.46) for $\frac{H'(N)}{H(N)}$ with $\Omega_{0\mathrm{m}} = 0.3$ and $\Omega_{0r} = 10^{-4}$. This ansatz, for $\frac{H'(N)}{H(N)}$, leads to the $w_{eff}(N)$ shown in Fig. 1. We have set up the system initially, on R_4, with $m = -0.5$. As seen in Fig. 2 and Fig. 3 the system follows the evolution of R_4 from the Radiation Era, to the Matter Era (M_4) and finally to the deSitter Era (Λ_4). We have checked that if we choose initial conditions not exactly coinciding with any of the other critical points then the system is captured by the R_4 "attractor" and follows the trajectory mentioned above, i.e. $(init) \rightarrow R_4 \rightarrow M_4 \rightarrow \Lambda_4$.

Finally, if the initial conditions of the system are set to be exactly on the top of other critical points then the system will end up on the Λ_1 critical point.

The choice of a constant m is justified by the fact that x_1 and x_3^2 are constants in each era (see Fig. 2). Also, the potential $U(\Phi)$ and coupling $F(\Phi)$ that are mostly used in the literature are power-laws or exponentials, which also give a constant m (see equation (2.26)). This feature will be obtained also by Noether symmetries, as we will discuss below.

III. RECONSTRUCTION OF $F(\Phi)$ AND $U(\Phi)$

A. Analytical Results

Our goal is now the reconstruction of the form of the potential $U(\Phi)$ and coupling $F(\Phi)$ corresponding to each one of the critical points of the system shown in Table I. Let us consider a critical point of the form $(\bar{x}_1, \bar{x}_2, \bar{x}_3^2, \bar{x}_4)$. Using (2.16), we find

$$F = F_0 e^{-\bar{x}_1 N} \tag{3.1}$$

where $F_0 = F(N = 0)$ is the present value of F. Using now eq. (2.18) we find

$$\begin{aligned} \Phi(N) &= -2\sqrt{6}\,\frac{\bar{x}_3}{\bar{x}_1}\,F_0^{1/2} e^{-\bar{x}_1 N/2} + C \\ &= 2\sqrt{6}\,\frac{\bar{x}_3}{\bar{x}_1}\,F_0^{1/2}\left(1 - e^{-\bar{x}_1 N/2}\right) + \Phi_0 \end{aligned} \tag{3.2}$$

where

$$C = \Phi_0 + 2\sqrt{6}\,\frac{\bar{x}_3}{\bar{x}_1}\,F_0^{1/2} \tag{3.3}$$

and $\Phi_0 \equiv \Phi(N = 0)$. Equations (3.1) and (3.2) allow us to eliminate N in favour of Φ

$$F(\Phi) = \frac{1}{24}\frac{\bar{x}_1^2}{\bar{x}_3^2}(\Phi - C)^2 \equiv \xi(\Phi - C)^2 \tag{3.4}$$

where $\xi \equiv \frac{1}{24}\frac{\bar{x}_1^2}{\bar{x}_3^2}$. The quadratic form of $F(\Phi)$ can be achieved in a completely different approach imposing Noether symmetry in the scalar-tensor Lagrangian.

From eq. (2.17), we have

$$U(N) = \bar{x}_2 \cdot 3F(N)H(N)^2 \tag{3.5}$$

Using now the input form of $H(N)$ (Eq.(2.45)), we find the dominant term of $H(N)$ in each era, that is

$$H(N)^2/H_0^2 = \begin{cases} \Omega_{0r} e^{-4N}, & \text{Rad. Era} \\ \Omega_{0\mathrm{m}} e^{-3N}, & \text{Mat. Era} \\ 1 - \Omega_{0r} - \Omega_{0\mathrm{m}}, & \text{dS Era} \end{cases}$$

Thus using eq.(3.2) we have $H(\Phi)$:

$$H(\Phi)^2/H_0^2 = \begin{cases} \frac{\Omega_{0r}}{F_0^{4/\bar{x}_1}}\left[\xi(\Phi - C)^2\right]^{4/\bar{x}_1}, & \text{Rad. Era} \\ \frac{\Omega_{0\mathrm{m}}}{F_0^{3/\bar{x}_1}}\left[\xi(\Phi - C)^2\right]^{3/\bar{x}_1}, & \text{Mat. Era} \\ 1 - \Omega_{0r} - \Omega_{\mathrm{m}}, & \text{dS Era} \end{cases}$$

and then we can express (3.5) in terms of Φ to get the relevant form

$$U(\Phi) = \lambda(\Phi - C)^{2+\alpha} \tag{3.6}$$

where,

$$\lambda = \begin{cases} 3\bar{x}_2 \frac{\Omega_{0r}}{F_0^{4/\bar{x}_1}} \xi^{1+4/\bar{x}_1}, & \text{Rad. Era} \\ 3\bar{x}_2 \frac{\Omega_{0\mathrm{m}}}{F_0^{3/\bar{x}_1}} \xi^{1+3/\bar{x}_1}, & \text{Mat. Era} \\ 3\bar{x}_2 \xi (1 - \Omega_{0r} - \Omega_{0\mathrm{m}}), & \text{dS Era} \end{cases}$$

and, $\alpha = \begin{cases} 8/\bar{x}_1, & \text{Rad. Era} \\ 6/\bar{x}_1, & \text{Mat. Era} \\ 0, & \text{dS Era} \end{cases}$

By using eqs. (2.26), (3.4) and (3.6), it is straightforward to find

$$2 + \alpha = 2m \tag{3.7}$$

so that Eq.(3.6) can be written as

$$U(\Phi) = \lambda(\Phi - C)^{2m} \tag{3.8}$$

Notice that even though the eigenvalue $\bar{x}_1$ changes in the sequence $R_4 \rightarrow M_4$, the exponent $2+\alpha$ remains constant.

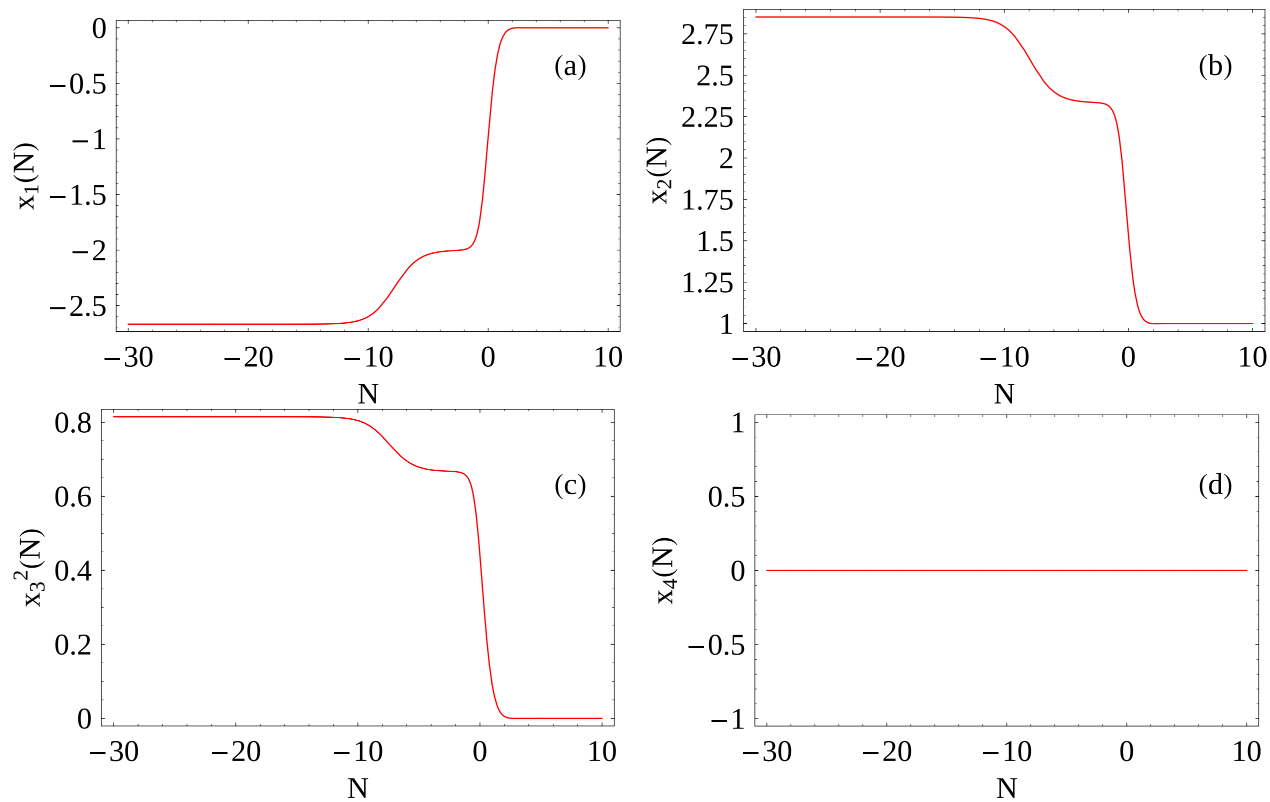

FIG. 2: The evolution of the variables $x_1(N)$, $x_2(N)$, $x_3^2(N)$ and $x_4(N)$. The system follows the evolution of the "attractor" through the three eras.

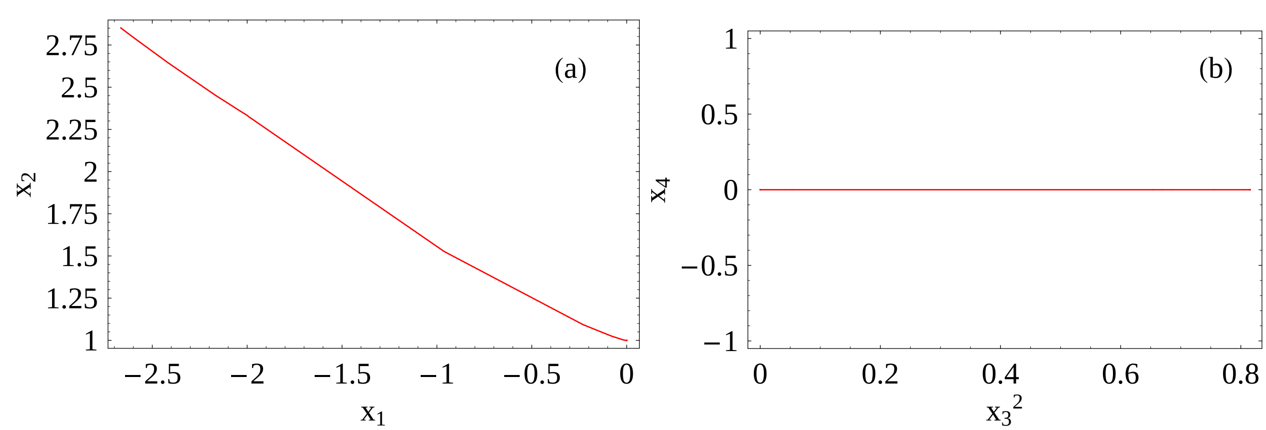

FIG. 3: The phase space trajectories on the $x_1 - x_2$ plane (Fig3a) and $x_3^2 - x_4$ plane (Fig3b).

Furthermore, since m is constant (see Eq.(3.7)) Eq.(2.26) allows to write U in terms of F, i.e.

$$U = cF^m \tag{3.9}$$

The above analysis is only valid when the parameters $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3^2$ are not equal to zero, as for example is the case for R_4 or when we have perturbed the initial conditions around a critical point. Otherwise, we have the following cases:

- $\bar{x}_2 = \bar{x}_3^2 = 0$, then $\Phi = \Phi_0 =$*const*, $F = F(N)$ given from eq. (3.1) and $U = 0$, as is the case for R_2 and R_3.
- $\bar{x}_1 = \bar{x}_3^2 = 0$ and we are in the deSitter era, then $\Phi = \Phi_0 =$*const*, $F = F_0$ and $U = U_0$, as is the case for Λ_4.
- $\bar{x}_1 = \bar{x}_2 = \bar{x}_3^2 = 0$, then $\Phi = \Phi_0 =$*const*, $F = F_0$ and $U = 0$, as is the case for R_5.

Note that if we were considering other (non-critical) points, not only the reconstruction would involve a time dependence, but we would not be able to reconstruct simultaneously $F(\Phi)$ and $U(\Phi)$ from the single function (1.1). Two arbitrary functions obviously need two observed functions to be reconstructed in the general case (see [14, 20, 21]). In our case, however, the fact that the reconstruction occurs on the critical points means that m is fixed to a constant and this closes the system of equations (2.22), (2.24), (2.25), (2.27) allowing us to proceed with the reconstruction numerically and analytically.

Also, it is easy to see that the reconstructed theories are merely Brans-Dicke theories with an additional potential U. If we define $F = \beta\phi_{BD} = \xi\Phi^2$ then $\Phi = \sqrt{\frac{\beta}{\xi}\phi_{BD}}$ and the Lagrangian (1.2) becomes

$$\mathcal{L} = \frac{\beta\phi_{BD}}{2}\, R - \frac{1}{2}\,\frac{\omega_{BD}}{\phi_{BD}}\, g^{\mu\nu}\partial_\mu\phi_{BD}\partial_\nu\phi_{BD} - U(\phi_{BD}) + \mathcal{L}_{\rm m} \tag{3.10}$$

where $\omega_{BD} \equiv \epsilon\frac{\beta}{\xi} = constant$.

A noteworthy feature of the above reconstruction scheme is that Eq.(3.9) along with Eq. (3.4) are exactly the conditions for the existence of a Noether symmetry in Scalar-Tensor theories as we discuss in what follows.

B. Noether Symmetries in Scalar-Tensor Gravity

1. *Generalities on the method*

Solutions for the Lagrangian (1.2) can be searched by the so called Noether Symmetry Approach [22]. This approach allows, in principle, to find out cyclic variables related to conserved quantities and then to reduce dynamics. Besides, the existence of symmetries fixes the forms of the coupling $F(\Phi)$, of the potential $U(\Phi)$ and gives the relation between them.

Let us give a quick summary of the approach for finite dimensional dynamical systems before the application to our specific problem.

Let $\mathcal{L}(q^i, \dot{q}^i)$ be a canonical, non-degenerate point-like Lagrangian in the configuration coordinates q^i (the "positions"), where

$$\frac{\partial\mathcal{L}}{\partial\lambda} = 0\,; \qquad {\rm det}H_{ij} \equiv {\rm det}\left|\left|\frac{\partial^2\mathcal{L}}{\partial\dot{q}^i\partial\dot{q}^j}\right|\right| \neq 0\,. \tag{3.11}$$

H_{ij} is the Hessian matrix related to $\mathcal{L}$. The dot indicates derivatives with respect to the affine parameter λ which, in general, corresponds to the time t. We are going to consider only transformations which are point-transformations. Any invertible and smooth transformation of the "positions" $Q^i = Q^i(\mathbf{q})$ induces a transformation on the "velocities" such that

$$\dot{Q}^i(\mathbf{q}) = \frac{\partial Q^i}{\partial q^j}\dot{q}^j\,. \tag{3.12}$$

The matrix $\mathcal{J} = ||\partial Q^i/\partial q^j||$ is the Jacobian of the transformation on the positions, and it is assumed to be nonzero. The Jacobian $\widetilde{\mathcal{J}}$ of the "induced" transformation is easily derived and it has to be $\mathcal{J} \neq 0 \rightarrow \widetilde{\mathcal{J}} \neq 0$. Usually, this condition is not satisfied in the whole space but only in the neighbor of a point. It is a *local transformation.* A point transformation $Q^i = Q^i(\mathbf{q})$ can depend on one (or more than one) parameter. In general, an infinitesimal point transformation is represented by a generic vector field acting on the space $\{q^i, \dot{q}^i\}$. The transformation induced by (3.12) is then represented by

$$\mathbf{X} = \alpha^i(\mathbf{q})\frac{\partial}{\partial q^i} + \left(\frac{d}{d\lambda}\alpha^i(\mathbf{q})\right)\frac{\partial}{\partial\dot{q}^i}\,. \tag{3.13}$$

$\mathbf{X}$ is called the "complete lift" of $\mathbf{X}$ [29]. A function $f(\mathbf{q}, \dot{\mathbf{q}})$ is invariant under the transformation $\mathbf{X}$ if

$$L_{\mathbf{X}}f \equiv \alpha^i(\mathbf{q})\frac{\partial f}{\partial q^i} + \left(\frac{d}{d\lambda}\alpha^i(\mathbf{q})\right)\frac{\partial f}{\partial\dot{q}^i} \,=\, 0\,, \tag{3.14}$$

where $L_{\mathbf{X}}f$ is the Lie derivative of f. In particular, if

$$L_{\mathbf{X}}\mathcal{L} = 0\,, \tag{3.15}$$

$\mathbf{X}$ is said to be a *symmetry* for the dynamics derived from the Lagrangian $\mathcal{L}$. To see how Noether's theorem and cyclic variables are related, let us consider a Lagrangian $\mathcal{L}$ and the related Euler-Lagrange equations

$$\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{q}^j} - \frac{\partial\mathcal{L}}{\partial q^j} = 0\,. \tag{3.16}$$

Let us consider also the vector field (3.13). By contracting (3.16) with the α^i's, one obtains

$$\alpha^j\left(\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{q}^j} - \frac{\partial\mathcal{L}}{\partial q^j}\right) = 0\,. \tag{3.17}$$

Being

$$\alpha^j \frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{q}^j} = \frac{d}{d\lambda}\left(\alpha^j\frac{\partial\mathcal{L}}{\partial\dot{q}^j}\right) - \left(\frac{d\alpha^j}{d\lambda}\right)\frac{\partial\mathcal{L}}{\partial\dot{q}^j}\,, \qquad (3.18)$$

from (3.17), we have

$$\frac{d}{d\lambda}\left(\alpha^i\frac{\partial\mathcal{L}}{\partial\dot{q}^i}\right) = L_{\mathbf{X}}\mathcal{L}\,. \qquad (3.19)$$

As a consequence, the *Noether Theorem* enunciates:

If $L_{\mathbf{X}}\mathcal{L} = 0$, the function

$$\Sigma_0 = \alpha^i\frac{\partial\mathcal{L}}{\partial\dot{q}^i}\,, \qquad (3.20)$$

is a constant of motion.

It is worth noting that Eq.(3.20) can be expressed independently of coordinates as a contraction of $\mathbf{X}$ by a Cartan one-form

$$\theta_{\mathcal{L}} \equiv \frac{\partial\mathcal{L}}{\partial\dot{q}^i}dq^i\ . \qquad (3.21)$$

Thus Eq.(3.20) can be written as

$$i_{\mathbf{X}}\theta_{\mathcal{L}} = \Sigma_0\ . \qquad (3.22)$$

where $i_{\mathbf{X}}$ is defined through the relation

$$i_{\mathbf{X}}dq^i = \alpha^i \qquad (3.23)$$

By a point–transformation, the vector field $\mathbf{X}$ becomes

$$\widetilde{\mathbf{X}} = (i_{\mathbf{X}}dQ^k)\frac{\partial}{\partial Q^k} + \left(\frac{d}{d\lambda}(i_{\mathbf{X}}dQ^k)\right)\frac{\partial}{\partial\dot{Q}^k}\ . \qquad (3.24)$$

$\widetilde{\mathbf{X}}$ is still the lift of a vector field defined on the "space of positions". If $\mathbf{X}$ is a symmetry and we choose a point transformation such that

$$i_{\mathbf{X}}dQ^1 = 1\,;\quad i_{\mathbf{X}}dQ^i = 0\quad i\neq 1\,, \qquad (3.25)$$

we get

$$\widetilde{\mathbf{X}} = \frac{\partial}{\partial Q^1}\,;\qquad \frac{\partial\mathcal{L}}{\partial Q^1} = 0\ . \qquad (3.26)$$

Thus Q^1 is a cyclic coordinate and the dynamics can be reduced [30, 31]. Clearly the change of coordinates defined by (3.25) is not unique. Usually a clever choice is very important. It is possible that more than one vector field $\mathbf{X}$ is found. In this case, more than one symmetry exists.

2. *The case of Scalar-Tensor Gravity*

The above method can be used to seek for solutions in the dynamics given by Lagrangian (1.2). In particular, for flat FRW metric, the field Lagrangian (1.2) reduces to the point-like Lagrangian

$$\mathcal{L} = -3a\dot{a}^2F - 3F_{,\Phi}\dot{\Phi}a^2\dot{a} + a^3\left(\frac{1}{2}\dot{\Phi}^2 - U(\Phi)\right) - Da^{-3(\gamma-1)}\,, \qquad (3.27)$$

where, for the sake of simplicity, we are considering only the scalar field case (the generalization to the phantom field case is obvious). The constant D is related to the perfect-fluid matter density, being $\rho_{\mathrm{m}} = D(a_0/a)^{3\gamma}$, where $1 \leq \gamma \leq 2$ defines the Zel'dovich range for the equation of state of standard matter. The above dynamical system (2.2), (2.3), (2.4) is immediately deduced considering the energy condition and the Euler-Lagrange equations for (3.27). In the case of standard dust matter, $\gamma = 1$, the last term in (3.27) reduces to an additive constant. Being $\{a, \Phi\}$ the configuration space of the system, the problem is 2D and then the infinitesimal generator of the Noether symmetry is

$$\mathbf{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial\Phi} + \dot{\alpha}\frac{\partial}{\partial\dot{a}} + \dot{\beta}\frac{\partial}{\partial\dot{\Phi}}, \qquad (3.28)$$

where α and β are functions depending on a and Φ, and

$$\dot{\alpha} \equiv \frac{\partial\alpha}{\partial a}\dot{a} + \frac{\partial\alpha}{\partial\Phi}\dot{\Phi}\quad;\quad \dot{\beta} \equiv \frac{\partial\beta}{\partial a}\dot{a} + \frac{\partial\beta}{\partial\Phi}\dot{\Phi}. \qquad (3.29)$$

The condition for the existence of a Noether symmetry is $L_{\mathbf{X}}\mathcal{L} = 0$. It, explicitly, gives an expression of second degree in $\dot{a}$ and $\dot{\Phi}$, whose coefficients are zero due to the fact they are considered to be linearly independent. Then this set of coefficients gives rise to the following system of partial differential equations [22],

$$\alpha + 2a\frac{\partial\alpha}{\partial a} + a^2\frac{\partial\beta}{\partial a}\frac{F_{,\Phi}}{F} + a\beta\frac{F_{,\Phi}}{F} = 0 \qquad (3.30)$$

$$\left(2\alpha + a\frac{\partial\alpha}{\partial a} + a\frac{\partial\beta}{\partial\Phi}\right)F_{,\Phi} + aF_{,\Phi\Phi}\beta + 2F\frac{\partial\alpha}{\partial\Phi} - \frac{a^2}{3}\frac{\partial\beta}{\partial a} = 0 \qquad (3.31)$$

$$3\alpha - 6F_{,\Phi}\frac{\partial\alpha}{\partial\Phi} + 2a\frac{\partial\beta}{\partial\Phi} = 0 \qquad (3.32)$$

$$\frac{U_{,\Phi}}{U} = -\frac{3\alpha}{a\beta} \qquad (3.33)$$

Equation (3.33) can be re-written in the form

$$\frac{U_{,\Phi}}{U} = m\cdot\frac{F_{,\Phi}}{F} \qquad (3.34)$$

where

$$m \equiv -\frac{3\alpha}{a\beta}\frac{F}{F_{,\Phi}} \qquad (3.35)$$

It is worth noting that Eq.(3.34) is a relation between the potential and the coupling and it exactly coincides

with (2.26). Solving the above system means to find out the explicit form of the set of functions $\{\alpha, \beta, F, U\}$. For this purpose we consider the separation of variables,

$$\alpha = A_1(a)A_2(\Phi) \tag{3.36}$$

and

$$\beta = B_1(a)B_2(\Phi) \tag{3.37}$$

Then from eq. (3.32) we get

$$\frac{B_1 a}{6A_1} = -\frac{A_2}{4B_2'} + \frac{A_2'F'}{2B_2'} = C \tag{3.38}$$

where C is a separation constant. The solution of (3.38) is simple and we get

$$A_1 = \frac{B_1 a}{6C} \tag{3.39}$$

and

$$B_2' = -\frac{A_2 - 2A_2'F'}{4C} \tag{3.40}$$

Using eqs. (3.36) and (3.37) on (3.30) we get

$$-\frac{a}{B_1}\frac{dB_1}{da} = \frac{3(A_2F + 2CB_2F')}{2(A_2F + 3CB_2F')} = -s \tag{3.41}$$

where s is a separation constant. Hence, from eqs. (3.39) and (3.41) we get

$$B_1 = Ba^s \tag{3.42}$$

and

$$A_1 = \frac{B}{6C}a^{s+1} \tag{3.43}$$

where B is a constant of integration. Also, from (3.41) we have

$$\frac{F'}{F} = -\frac{2s+3}{6C(s+1)}\frac{A_2}{B_2} \tag{3.44}$$

Using eqs. (3.42), (3.43) and (3.32) yields

$$2FA_2' + (A_2(s+3) + 6CB_2')F' - 2B_2C(s - 3F'') = 0 \tag{3.45}$$

Now we are left with three equations (3.40), (3.44) and (3.45) to solve for the three unknown functions A_2, B_2 and F. Using eqs. (3.40) and (3.44) for B_2 and B_2' on eq. (3.45) we get

$$FA_2' + \frac{F'}{4}(A_2(2s+3) + 6A_2'F') + \frac{A_2F(2s+3)(s-3F'')}{6(s+1)F'} = 0 \tag{3.46}$$

Also, using (3.44) in (3.40) yields

$$\begin{aligned} &2A_2'F'\left(3(s+1)(F')^2 + F(2s+3)\right) + \\ &+A_2\left((s+3)(F')^2 - 2F(2s+3)F''\right) = 0 \end{aligned} \tag{3.47}$$

Now, we can use equations (3.46) and (3.47) to eliminate A_2 and A_2' in favor of F,

$$\begin{aligned} F'' &= \frac{3s(s+1)(s+2)F'^4}{(2s+3)F^2} + \\ &+ \frac{(s+1)\left(8s^2+16s+3\right)F'^2}{2(2s+3)F} + \frac{s(2s+3)}{3} \end{aligned} \tag{3.48}$$

Equation (3.48) is nonlinear and its complete solution is an elliptical integral of second kind which is not simple to handle. However, an exact solution can be found to be of the form

$$F = \xi(\Phi - \Phi_0)^2 \tag{3.49}$$

Using the ansatz (3.49) in (3.48) we get that $\xi = -\frac{(2s+3)^2}{24(s+1)(s+2)}$ or $\xi = -\frac{1}{6}$, with the latter corresponding to the conformal coupling [32]. Note also that the conformal coupling $\xi = -1/6$ corresponds to a model where there is actually no scalar degree of freedom despite the fact that there seems to exist one in the parametrization (1.2) (see [33]). The free parameter s has a physical meaning since it is connected to the ratio of critical points $\bar{x}_1$ and $\bar{x}_3$ and to the coupling. For this form of $F(\Phi)$ we can now determine A_2 from (3.46) or (3.47) and B_2 from (3.44) and finally arrive at a solution for α and β. For the two values of ξ and eqs. (3.46) and (3.47) we get three degenerate solutions for α and β, ie they correspond to the same form of the potential U (see eq.(3.33)). The solutions are

$$\begin{aligned} \alpha_1 &= \frac{a^{s+1}AB(\Phi-\Phi_0)^{\frac{2s(s+2)}{2s+3}}}{6C} \\ \beta_1 &= \frac{a^sAB(2s+3)(\Phi-\Phi_0)^{\frac{2s(s+2)}{2s+3}+1}}{12C(s+1)} \end{aligned}$$

$$\begin{aligned} \alpha_2 &= \frac{a^{s+1}AB(\Phi-\Phi_0)^{\frac{s(2s+3)}{2(s+1)}}}{12C(s+1)} \\ \beta_2 &= \frac{a^sAB(2s+3)(\Phi-\Phi_0)^{\frac{s(2s+3)}{2(s+1)}+1}}{12C(s+1)} \end{aligned}$$

$$\alpha_3 = \frac{a^{s+1}AB(\Phi_0-\Phi)^s}{6C}$$
$$\beta_3 = \frac{a^s AB(2s+3)(\Phi_0-\Phi)^{s+1}}{12C(s+1)}$$

It is easy to show that for all three cases we have

$$m(s) = \frac{3(s+1)}{2s+3} \tag{3.50}$$

and that from eq. (3.34) we get

$$U(\Phi) = U_0(\Phi-\Phi_0)^{\frac{6(s+1)}{2s+3}} = U_0(\Phi-\Phi_0)^{2m(s)} \tag{3.51}$$

where U_0 is a constant determining the scale of the potential and is not directly measurable, but can be rewritten in terms of observable parameters like H_0, q_0, $\Omega_{\rm m}$. A noteworthy feature of eq. (3.51) is that it exactly coincides with (3.8), thus hinting towards a non-trivial physical content in this class of scalar-tensor Lagrangians.

In order to find the solution to the field equations we need to find the value of the constant of motion Σ_0 from eq. (3.20). This is,

$$\Sigma_0 = \frac{b(2s+3)^2}{(s+1)^2(s+2)(s+3)} \frac{d\left(a^{s+3}(\Phi-\Phi_0)^{\frac{2s(s+2)}{2s+3}+2}\right)}{dt} \tag{3.52}$$

where $b = -\frac{AB}{48C}$. Integration of (3.52) yields

$$\Phi-\Phi_0 = a^{-\frac{2s+3}{2s+2}} c^{1+\frac{3}{2s}} t^{\frac{2s+3}{2s^2+8s+6}} \tag{3.53}$$

where $c = \left(\frac{(s+1)^2(s^2+5s+6)\Sigma_0}{b(2s+3)^2}\right)^{\frac{s}{(s+1)(s+3)}}$.

Plugging (3.51) and (3.53) into (2.2) we can get $a(t)$ and $\Phi(t)$:

$$a(t) = t^{\frac{s+2}{s+3}}\left(\frac{8cs(s+1)(s+2)(s+3)^2U_0t^{\frac{s+6}{s+3}}}{(s+6)(2s+3)^2} + a_0\right)^{1+\frac{1}{s}} \tag{3.54}$$

and

$$(\Phi(t)-\Phi_0)^2 = c^{2+\frac{3}{s}} t^{-\frac{2s+3}{s+3}} \cdot \left(\frac{8cs(s+1)(s+2)(s+3)^2U_0t^{\frac{s+6}{s+3}}}{(s+6)(2s+3)^2} + a_0\right)^{-2-\frac{3}{s}} \tag{3.55}$$

where a_0 is an integration constant.

Due to the structure of the above general solution, the cases $s = 0, -1, -3/2, -2$ and $s = -3$ have to be considered apart. The solutions for $s = 0$ and $s = -3$ correspond to the minimal coupling where $F = F_0$ and $U = \Lambda$ and to the quartic potential case where $F \sim \Phi^2$ and $U \sim \Phi^4$. In these situations, the solutions assume oscillating or exponential behavior (for a discussion see [22, 23, 34]).

C. Numerical results

As final step, in order to confirm the validity of our analysis, we perform a numerical evolution of the dynamical system to compare the form of the potential U and coupling F of eqs.(3.4) and (3.8) (found also by the Noether Symmetry Approach), with the corresponding form obtained from the numerical analysis. The steps involved in the comparison are the following:

- Numerically solve the dynamical system (2.22), (2.24), (2.25), (2.27) and obtain $x_1(N)$, $x_2(N)$, $x_3^2(N)$ and $x_4(N)$. Integrate equation (2.16) to get
$$F(N) = F_0 e^{-\int_{N_{min}}^{N} x_1(N')dN'} \tag{3.56}$$
and use eq. (2.18) in the form
$$\Phi'(N)^2 = 6x_3^2(N)F(N) \tag{3.57}$$
to obtain $\Phi(N)$. The resulting form of $F(\Phi)$ in both the numerical (red continuous line) Eq.(3.56) and its analytical approximation (blue dotted line) Eq.(3.4) is shown in Fig. 4.
- Use equation (2.17) to obtain $U(N)$ numerically (red continuous line in Fig. 5)
$$U(N) = 3x_2(N)F(N)H(N)^2 \tag{3.58}$$
and compare with the analytical form (blue dotted line in Fig. 5) of Eq.(3.8).

In the methodology mentioned above we used the exact coefficients $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3^2$ and $\bar{x}_4$ from Table I in each era for the analytic forms (3.4) and (3.8). Also, the initial conditions used in the numerical evolution were $F(N=-30) = \Phi(N=-30) = 1$. As another test, we fitted the numerically obtained $F(N)$ and $U(N)$ from eqs.(3.56) and (3.58) respectively to obtain the coefficients of the analytic forms (3.4) and (3.8) and found that the results were in good agreement (see Figs. 4 and 5 the green long-dashed lines). A noteworthy feature of Figs. 4 and 5 is a small plateau that appears during the deSitter era (see the zoomed region in Fig. 4). The reason for the existence of the plateau is that as the system evolves towards the deSitter era the coupling $F(\Phi)$ "freezes", since $x_1(N_{dS}) \to 0$ (see eq. (2.16) and Fig. 2a), much faster than the field Φ.

IV. CONCLUSIONS AND OUTLOOK

We have reconstructed the form of the gravitational coupling $F(\Phi)$ and the potential $U(\Phi)$ of scalar-tensor quintessence by demanding that it reproduces a ΛCDM cosmic history through the radiation ($w_{eff} = \frac{1}{3}$), matter ($w_{eff} = 0$), and deSitter ($w_{eff} = -1$) eras. We have found that apart from the usual general relativistic solution with a constant coupling $F(\Phi) = F_0$ and potential

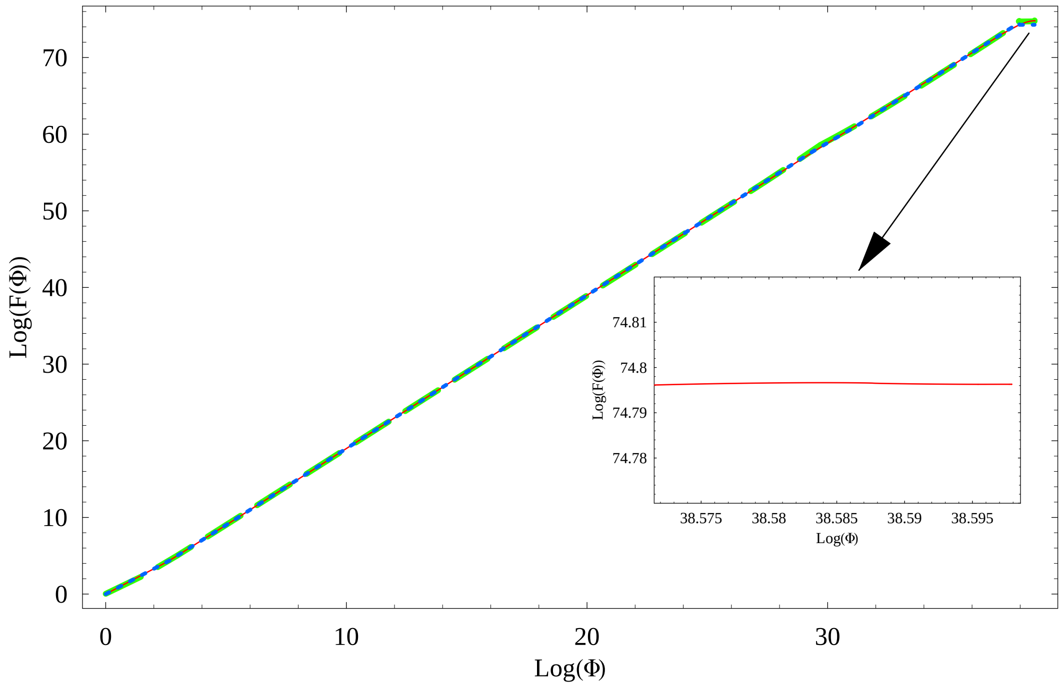

FIG. 4: The form of $log(F(\Phi))$ in the numerical reconstruction (red continuous line), its analytical approximation (blue dotted line) and a fit of the numerical reconstruction using Eq.(3.56) (green long-dashed line). The agreement between the three approaches is very good. The reason for the existence of the small plateau, see the zoomed region, is that as the system evolves towards the deSitter era the potential $F(\Phi)$ "freezes" much faster than the field Φ.

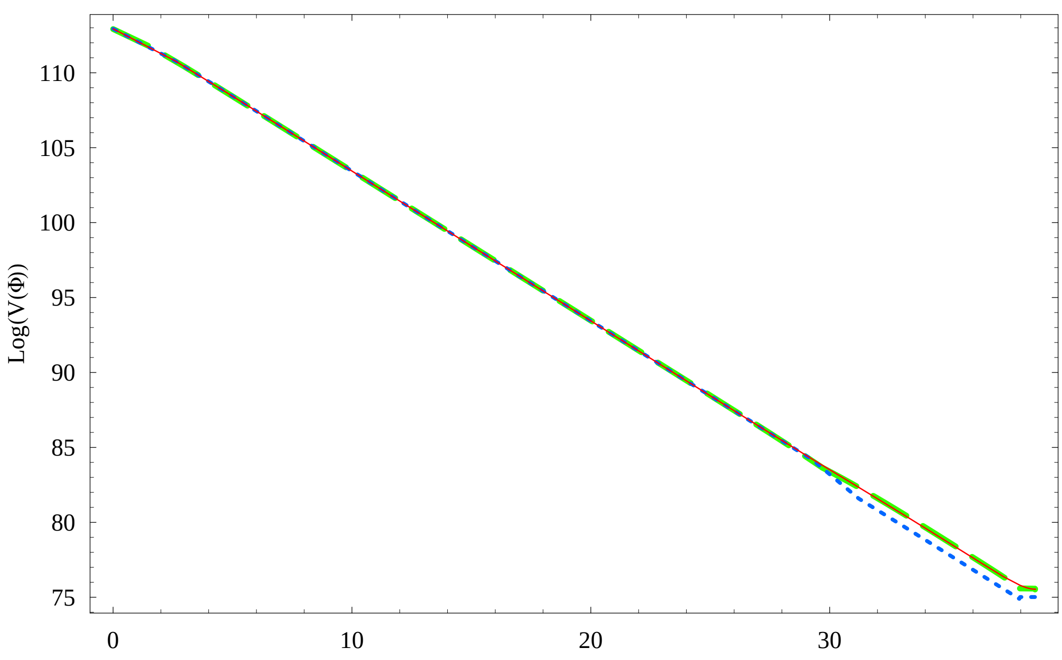

FIG. 5: The form of $log(V(\Phi))$ in the numerical reconstruction (red continuous line), its analytical approximation (blue dotted line) and a fit of the numerical reconstruction (green long-dashed line). The agreement between the three approaches is very good. The potential exhibits a small plateau in the deSitter era for the same reason as $F(\Phi)$ (see caption of Fig. 4)

$U(\Phi) = U_0$ (corresponding to Newton and cosmological constants), there is another consistent solution which reproduces the same cosmic history. According to this solution

$$F(\Phi) \;=\; \xi(\Phi - C)^2 \tag{4.1}$$

$$U(\Phi) \;\sim\; F(\Phi)^m \tag{4.2}$$

where m, C are arbitrary constants (m however is negative). In this new solution the 'radiation', 'matter' and 'deSitter' expansion rates, for the 'attractor' trajectory shown in Figs 2 and 3, is dominated by dark gravity through all epochs. This is indeed a potential problem for this type of trajectories but it could also be a potential blessing since this type of solutions have the correct expansion rate at all epochs without the use of dark matter or dark energy. A proper test of these models for detailed comparison with observations would require analysis of large scale structure formation (analysis of evolution of *perturbations*). Such an analysis is beyond the aims of our present analysis but it is an interesting extension of this project.

We should also stress that not only the attractors but all the fixed points we found are potentially interesting because as discussed above a more physical stability analysis would fix $F(N)$ and allow $H(N)$ to vary thus introducing and eliminating instability modes. Thus the physically interesting part of our analysis is the actual values of the fixed points and not their stability which could change if $H(N)$ were allowed to vary. On the other hand, going back to the considerations developed in Sect.II regarding the number of parameters beside m, we have to stress again that if we had used the parameter $n \sim F_{,\Phi}/F$ in our analysis (thus fixing $F(\Phi)$ and allowing $H(N)$ to vary), the autonomous system would be very different and, depending on the values of n, the stability and the form of $H(N)$ would correspondingly vary. An alternative approach could be to fix F and U, as in Eqs.(4.1) and (4.2), and allow $H(N)$ to vary about a the ΛCDM background, along the lines of Ref. [26]. Very likely, this approach could provide a comprehensive stability analysis of the ΛCDM model since it involves also the proper fluctuation modes of $H(N)$. This extension is out of the lines of this work and will be faced in a forthcoming paper.

Another point is that phantom behavior can be easily realized in scalar tensor theories, see Refs [11, 35] for a discussion, and this is an attractive feature of these theories. In fact we could have chosen to reconstruct different forms of $H(N)$ giving late time phantom behavior thus deriving different forms of $F(N)$ at late times with different fixed points. This late time reconstruction has been undertaken in Refs. [17] and [36]. However, clearly the early times behavior of the reconstructed $F(N)$ would be unchanged even in the phantom case. Fixing $F(\Phi)$ could also lead to phantom behavior but we would have to guess a proper form of F.

A completely independent way that can also lead to the form of the new solution presented here is obtained by imposing maximal Noether symmetry on the scalar-tensor Lagrangian. We have demonstrated that imposing such a symmetry leads uniquely to exactly the same form of potentials as (4.1) and (4.2). It also leads to a conserved charge Σ_0 which allows the derivation of exact solutions for the evolution of the scale factor $a(t)$ and the scalar field $\Phi(t)$.

This intriguing coincidence of the two approaches hints towards a non-trivial physical content in this class of scalar-tensor Lagrangians. It is therefore important to study the evolution of cosmological perturbations in this class of models in order to test them using Large Scale Structure and CMB observations.

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